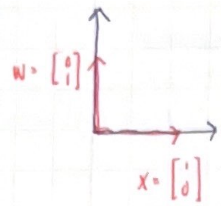
602 NOTES (SPANNING SETS & LINEAR DEPENDENCY)

- vectors span if they can create any other vector in the space (set of all linear combinations)
- vectors are independent if the only way to make them equal is to multiply by 0.



- w and x are linearly independent because they cannot be written in terms of each other unless $c\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where $c = 0$.

  otherwise dependent.

- $\cancel{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$ $u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $w^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

- u, v, w are independent. No combination except $0u + 0v + 0w = 0$ gives $b = 0$
- u, v, w* are dependent. Other combinations like $u + v + w^*$ give $b = 0$

- $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ span $\hat{\mathbb{R}}^2$ because we can make any vector in $\mathbb{R}^2$ with a linear combination of the two. Ex: $12\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$ any vector!!!

- If vectors are independent then they span. Independence = Span ✗ wrong
- vectors are <u>basis</u> if they span and are linearly independent.

- The vectors (1,1) and (0,0) are dependent because of the zero vector (pg. 166)
- In $\mathbb{R}^2$, any three vectors (a,b) (c,d) and (e,f) are dependent (pg. 166)
  - ↳ Every vector v in the space is a combination of the basis vectors, because they span the space.
  - the combination that produces v is unique because the basis vectors are independent.

- the <u>dimension</u> of a space is the number of vectors in every basis.

Prove that if a set has the $\vec{0}v$ it is linearly dependent: you can have any coefficient in front of the 0 vector, ~~you~~ and still get a zero linear combination.

SPANNING SET: given a linear space V, a non empty set of vectors $S = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is called a spanning set of V iff any vector $\vec{x} \in V$ can be written as a linear combination of vectors in S.

BASIS: there are many bases in a given linear space V. All of these bases have the exact same number of vectors. The number of vectors in (any) basis of V is called the <u>dimension</u> of V.

continued ⟹

SUBSPACES

* the span of n vectors is a valid subspace of $\mathbb{R}^n$

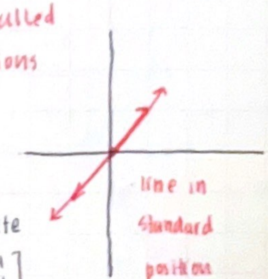
V subspace of $\mathbb{R}^n$

- V contains $0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$
- $\vec{x}$ in V = $c\vec{x}$ in V (for some vector x in set V, $\vec{x}$ multiplied by scalar is still in V)
  (closure under scalar multiplication)
- $\vec{a}$ in V, $\vec{b}$ in V: $\vec{a}+\vec{b}$ in V (for two arbitrary elements in V, the sum of them is still in V)
  (closure under addition)
- A subspace implies all three conditions are met

EX: $V = \{0\} = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$ V is subspace of $\mathbb{R}^3$

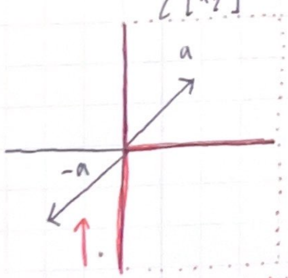
- contains $\vec{0}$ vector
- $c\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ closed under multiplication
- $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ closed under addition

EX: $U = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ all of the scaled up and scaled down versions



- $0\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ closed under 0 vector ✓
- $c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ any constant will create a scaled version of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so closed under multiplication ✓
- $c_1\overset{\vec{a}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + c_2\overset{\vec{b}}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = (c_1+c_2)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
  another scalar multiplied by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so closed under addition.

EX: $S = \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, x_1 \geq 0\right\}$ is S subspace of $\mathbb{R}^2$? NO!



- contains 0 vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$
  - a and c are greater than 0 so $a+c$ is ≥ 0
  - closed under addition
- $-1\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix}$
  - first component must be ≥ 0
  - not closed under scalar multiplication

* EX: is $\text{span}(v_1, v_2, v_3)$ = valid subspace in $\mathbb{R}^n$?
  $U = \text{span}(v_1, v_2, v_3)$

- $0v_1 + 0v_2 + 0v_3 = 0$, contains zero vector ✓
- $\vec{x} = c_1v_1 + c_2v_2 + c_3v_3$ linear combination of one vector
  $ax = \underbrace{ac_1}v_1 + \underbrace{ac_2}v_2 + \underbrace{ac_3}v_3$
  arbitrary constants
  $c_4v_1 + c_5v_2 + c_6v_3$ another linear combination of the vector
  so closed under multiplication ✓
- $\vec{y} = d_1v_1 + d_2v_2 + d_3v_3$ another vector
  $\vec{x}+\vec{y} = (c_1+d_1)v_1 + (c_2+d_2)v_2 + (c_3+d_3)v_3$
  - another linear combination of v_1, v_2, v_3 with arbitrary constants
  - closed under multiplication ✓

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

~ lin. alg. ~ october

without loss of generality, $(\forall)$ U, V, W finite dim subspace $\Rightarrow$

$\dim(U) + \dim(V) + [\dim(W) - \dim(U+V+W)] = \dim(U) + \dim(V) + [\dim\{(U+V) \cap W\} - \dim(U+V)]$

because $\underline{\dim(W) - \dim(U+W) = \dim(U \cap W) - \dim(U)}$

$U \Rightarrow (U+V)$

$\geq \dim(U) + \dim(V) - \dim(U+V)$

$= \dim(U \cap V)$

- the sum of two subspaces W_1 and W_2 is a direct sum iff $W_1 \cap W_2 = \{\vec{0_V}\}$

* DEF OF LINEAR TRANSFORMATION

- let V and W be two linear spaces. A function $f: V \to W$ is linear iff

$f(\alpha\vec{x} + \beta\vec{y}) = \alpha f(\vec{x}) + \beta f(\vec{y})$ $(\forall)$ $\vec{x}, \vec{y} \in V$, $(\forall)$ $\alpha, \beta \in \mathbb{R}$ — should hold true for any number of vectors

LECTURE 04 LINEAR FUNCTIONS!

Base case:

$f(\sum_1 \alpha x) = \sum_1 \alpha f(x)$

for one value: $f(\alpha x) = \alpha f(x)$

$f(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3) = \alpha_1 f(x_1) + \alpha_2 f(x_2) + \alpha_3 f(x_3)$

LINEAR SPACES

$f(\sum_1 \alpha x) = \sum_1 \alpha f(x)$

starting from left hand side

\# $f(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n)$

$= f(\alpha_1 x_1) + f(\alpha_2 x_2) + \ldots + f(\alpha_t x_t)$ definition of LT

$= \alpha_1 f(x_1) + \alpha_2 f(x_2) + \ldots + \alpha_t f(x_t)$ definition of LT

Prove $f(\vec{0_V}) = \vec{0_W}$ $\quad x = y$, $\beta = -1$, $\alpha = 1$

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \to$ def of LT

$f(x - x) = f(x) - f(x)$

$f(0) = f(0) - f(0)$

$f(\vec{0_V}) = \vec{0_W}$

$f: V \to W$ is linear function

- prove null f is subspace of V
  - show closed under addition and multiplication
  - $A, B \in$ null(f) then — condition 2

    $f(A) = 0$ and $f(B) = 0$. Hence $f(A) + f(B) = 0 + 0 = 0$. However,

* f is linear transformation so, $f(A) + f(B) = f(A+B)$. So, $f(A+B) = 0$

By def of Null, $A + B \in$ Null(f).

Thus, Null(f) is closed under addition.

  - $A \in$ Null(f) and $k \in \mathbb{R}$. Since

$A \in$ Null(f), $f(A) = 0$. then $k f(A) = k0 = 0$.

Since f is a linear transformation

$k f(A) = f(kA)$, so $f(kA) = 0$ and $kA \in$ Null(f). therefore Null(f) is closed under scalar multiplication.

{ SUBSPACE PROPERTIES OF NULL: }

1) Null(A) always contains the zero vector, since $A0 = 0$.
2) If $x \in$ Null(A) and $y \in$ Null(A) then $x + y \in$ Null(A)
3) if $x \in$ Null(A) and c is a scalar ~~c~~$\in k$ then $cx \in$ Null(A) since

$A(cx) = cA(x) = c0 = 0$

602 11/08/21

eigenvector cannot be 0:

$$\begin{bmatrix} A \in \mathbb{R}^{n \times n} \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix} = \lambda \begin{bmatrix} \vec{x} \end{bmatrix}$$ if $\vec{x}$ is 0 then you get the identity matrix...

Questions:
- purpose of determinant?
- → purpose of eigenvalues/eigenvectors is to describe natural language of a system
- purpose of characteristic polynomial?
  ↳ to find eigenvalues

Determinants:
- $\det(A) = \det(A^T)$
- A is invertible $\Leftrightarrow \det(A) \neq 0$
- $\det(A \cdot B) = \det(A)\ \det(B)$ (Binet-Cauchy rule)

- if any of A or B is singular, then so is (AB) (HW)
- $\det(A^{-1}) = \frac{1}{\det(A)}$ (HW)
  ↳ a singular matrix is a matrix whose determinant is 0 and has no inverse

↓

$\det(A^{-1})\det(A) = 1$

- If A is triangular, then $\det(A) = a_{11} \cdot a_{22} \ldots a_{nn}$ (diagonals)

Theorem: If $AB = BA$ then A and B share a common eigenvector. PROVE!

THE CHARACTERISTIC POLYNOMIAL:

$$\det(\lambda I_n - A) = \det\left[\begin{bmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda \end{bmatrix} - \begin{bmatrix} a_1 & \cdots & a_n \\ \vdots & & \\ \vdots & & a_{nn} \end{bmatrix}\right] = \det\begin{bmatrix} \lambda a_1 & \cdots & -a_n \\ \vdots & \lambda a_2 & \\ -a_{n_1} & & \lambda a_n \end{bmatrix}$$

is always a polynomial of degree exactly n in the undeterminate λ.

$\det(\lambda I_n - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \ldots + \alpha_{n-1}\lambda + \alpha_n$ is called the characteristic polynomial of A.

THEOREM: If λ_i is an eigenvalue of A then it is a root of $P_A(\lambda)$, the characteristic polynomial of A.

SIMILARITY TRANSFORMATION:

let $A \in \mathbb{R}^{n \times n}$ and $T \in \mathbb{R}^{n \times n}$, non singular. Then $(T^{-1}AT) \in \mathbb{R}^{n \times n}$ is called a similarity transformation of matrix A.

THEOREM: similarity transformation preserves the set of eigenvalues of A, that is, given $A \in \mathbb{R}^{n \times n}$ and any $T \in \mathbb{R}^{n \times n}$, invertible, A and $T^{-1}AT$ have the same eigenvalues.

PROOF: A and $T^{-1}AT$ have the same characteristic polynomial... don't feel like writing any more.

MIDTERM REVIEW

① compute all the matrices $A \in \mathbb{R}^{2\times 2}$, which have the following eigenvectors:

$\vec{v}_1 = \begin{bmatrix} \pi \\ -\pi \end{bmatrix}$ & $\vec{v}_2 = \begin{bmatrix} 200 \\ 100 \end{bmatrix}$

FACT: If $\vec{x}$ is an eigenvector of A, then $\alpha\vec{x}$, $(\forall)\alpha \in \mathbb{R}$ is also an eigenvector of A.

PROOF: $A\vec{x} = \lambda\vec{x} \Rightarrow A(\alpha\vec{x}) = \alpha \cdot A\vec{x} = \alpha\lambda\vec{x} = \lambda(\alpha\vec{x})$

$\vec{v}_1 = \pi\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ & $\vec{v}_2 = 100\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is also an eigenvector of A associated with eigenvalue λ (let)

$A\vec{x} = \lambda\vec{x} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \lambda \\ -\lambda \end{bmatrix}$ $\quad \begin{matrix} a_{11} - a_{12} = \lambda \\ a_{21} - a_{22} = -\lambda \end{matrix} \Rightarrow \{a_{11} - a_{12} = a_{22} - a_{21}\}$ ①

Similarly, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is also an eigenvector of A associated with eigenvalue μ.

$A\vec{x} = \mu\vec{x} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \mu\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\mu \\ \mu \end{bmatrix}$ $\quad \begin{matrix} 2a_{11} + a_{12} = 2\mu \\ 2a_{21} + a_{22} = \mu \end{matrix} \Rightarrow \{2a_{11} + a_{12} = 4a_{21} + 2a_{22}\}$ ②

let $a_{22} = l$, $a_{21} = k$, $(\forall)$, $l \in \mathbb{R}$

① $\Rightarrow a_{11} - a_{12} = l - k$ $\quad \Rightarrow$ ①+② $\Rightarrow 3a_{11} = 3k + 3l$ $\quad$ from ① $\Rightarrow (k+l) - a_{12} = l - k$

② $\Rightarrow 2a_{11} + a_{12} = 4k + 2l$ $\quad \Rightarrow a_{11} = (k+l)$ $\quad \Rightarrow a_{12} = 2k$

$A = \begin{bmatrix} l+k & 2k \\ k & l \end{bmatrix}$ $(\forall)\, l, k \in \mathbb{R}$ this is for all matrices. EX: $A = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$

② let $A = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & 1 & 1 \end{bmatrix}$, find $S \in \mathbb{R}^{3\times 3}$ s.t. $S^{-1}AS$ is diagonal.

First, find eigenvalues of A: $\det(A - \lambda I) = 0 \Rightarrow -\lambda[(3-\lambda)(1-\lambda) - 3] + 2(1 - \lambda + 3) - 3(1 + 3 - \lambda) = 0$

$\Rightarrow -\lambda(\lambda^2 - 4\lambda) - 1(4\lambda) = 0$

$\Rightarrow (\lambda - 4)(-\lambda^2 + 1) = 0$

$\Rightarrow \lambda = 4,\ \lambda^2 = 1,\ \lambda = 1, -1$

(CHECK PHOTOS)

For $\lambda_1 = 4$: (can pick $x_3 = 1$ arbitrary)

$\vec{x} = \begin{bmatrix} 3/2 \\ -3/2 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix} \rightarrow$ 1st eigenvector. ignore scalar!

For $\lambda_2 = 1$: $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ (check photos)

For $\lambda_3 = -1$: $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ (check photos)

$S = \begin{bmatrix} 3 & 1 & 1 \\ -3 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$ $\quad S^{-1}AS = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ check with $S^{-1} = \frac{1}{10}\begin{bmatrix} 5 & 5 & 0 \\ 5 & 1 & 6 \\ 0 & 2 & 2 \end{bmatrix}$

$S^{-1}AS = \frac{1}{10}\begin{bmatrix} 5 & 5 & 0 \\ 5 & 1 & 6 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ -3 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 5 & 5 \\ 5 & 1 \\ 0 & 2 \end{bmatrix}$ ...

#2 study for midterm ♡

you have to check and confirm you are correct.

check photos he changed the order of lambda

EE 602 NOTES

- a singular matrix is a matrix whose determinant is 0. and thus has no inverse.
- rank = number of independent columns from RREF
- $\lambda_1 + \lambda_2 + \ldots + \lambda_n = \text{trace} = a_{11} + a_{22} + \ldots + a_{nn}$
- invertible matrix is a matrix s.t. $A = A^{-1}$ and $A^2 = I$

Eigenvalues & Eigenvectors:

① eigenvector x lies along the same line as Ax. → λ is an eigenvalue of A iff $(A-\lambda I)$ is singular
$Ax = \lambda x$ The eigenvalue is λ

② If $Ax = \lambda x$ then $A^2 x = \lambda^2 x$ and $A^{-1}x = \lambda^{-1}x$ ⟶ when A is squared, the eigenvectors stay the same. The eigenvalues are squared.. And if x is eigenvector then $(\forall)\alpha$, αx is also eigenvector (pg. 92)
and $(A + cI)x = (\lambda + c)x$

③ If $Ax = \lambda x$ then $(A - \lambda I)x = 0$ and $A - \lambda I$ is singular and $\det(A - \lambda I) = 0$ with n eigenvalues → columns of $(A - \lambda I)$ are linearly independent

④ check λ's by $\det A = (\lambda_1)(\lambda_2)\ldots(\lambda_n)$ and diagonal sum = sum of λ's

- If $Ax = 0x$ then eigenvector x is in the nullspace
- If A is identity, every vector has $Ax = x$. All vectors are eigenvectors of I.

L.06:
- If $A\vec{x} = \lambda\vec{x}$ prove that $\vec{x}$ is also an eigenvector of A^n.

$A^n\hat{x} = (A^{n-1}A\hat{x}) = A^{n-1}(A\hat{x}) = A^{n-1}(\lambda\hat{x}) =$
$\lambda A^{n-1}x = \lambda^2 A^{n-2}\hat{x} \ldots \lambda^n A^0 \hat{x} = \lambda^n \hat{x}$
Therefore, $A^n\hat{x} = \lambda^n\hat{x}$

- If $\hat{x}$ is an eigenvector of A corresponding to eigenvalue λ then so is $(\alpha\hat{x})$ $(\forall)\alpha \in \mathbb{R}$

If $A\hat{x} = \lambda x$ then $A(\alpha\hat{x}) = \underbrace{\alpha(A\hat{x})}_{\text{linearity of A}} = \alpha(\lambda\hat{x}) = \lambda(\alpha\hat{x})$

$A(\alpha\hat{x}) = \lambda(\alpha\hat{x})$ which means that $\alpha\hat{x}$ is still an eigenvector.

Solve for eigenvalue & eigenvector:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} \Rightarrow (1-\lambda)(3-\lambda) - 4(2) = 0$$
$$(\lambda - 5)(\lambda + 1)$$
$$\lambda = 5, -1$$

For $\lambda = 5$:

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 5\begin{bmatrix} x \\ y \end{bmatrix}$$

$x + 4y = 5x$
$2x + 3y = 5y$
(solve for x and y)
(choose free variable = 1)
if $x = 1$ then eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $x = 1$ is "special number"

HOMEWORK 05:

- $\lambda = 0$ means matrix is singular
- A and A^{-1} has same eigenvectors with A^{-1} having eigenvalues $1/\lambda$.
- if needed do rref to find free vars.

Determinants:

- $\det(A) = \det(A^T)$
- $\det(AB) = \det(A)\det(B)$
- If $AB = BA$, then A and B share a common eigenvector matrix X

If the same X diagonalizes both $A = X\Lambda_1 X^{-1}$ and $B = X\Lambda_2$ we can multiply in any order.
$AB = X\Lambda_1 X^{-1} X\Lambda_2 X^{-1} = X\Lambda_1\Lambda_2 X^{-1}$
$BA = X\Lambda_2 X^{-1} X\Lambda_1 X^{-1} = X\Lambda_2\Lambda_1 X^{-1}$. Since $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$ (diagonal matrices commute) we have $AB = BA$

DIAGONALIZATION OF A MATRIX:

1) The columns of $AX = X\Lambda$ are $Ax_k = \lambda_k x_k$ the eigenvalue matrix Λ is diagonal.
2) n independent eigenvectors in X diagonalize A.
$A = X\Lambda X^{-1}$ and $\Lambda = X^{-1}AX$
3) the eigenvector matrix X diagonalizes all powers A^k: $A^k = X\Lambda^k X^{-1}$
4) no equal eigenvalues ⇒ X invertible A is diagonalized
equal eigenvalues ⇒ A might have too few independent eigenvectors. X^{-1} fails.

similarity transformation

CHARACTERISTIC POLYNOMIAL:

- when solving for eigenvalues we do $\det(A - \lambda I) = 0$. the polynomial equation this process yields is the characteristic polynomial
  - a scalar $\lambda \in \mathbb{R}$ is called an eigenvalue if $\exists$ a nonzero vector x such that $Tx = \lambda x$. If x is nonzero then with $x = 0$ every scalar $\lambda \in \mathbb{R}$ satisfies the equation $Tx = \lambda x$.

NULLSPACE OF A:

① nullspace $N(A) \in \mathbb{R}^n$ contains all solutions x to $Ax=0$ this includes $x=0$.

② Every matrix with $m<n$ has nonzero solutions to $Ax=0$ in its nullspace

- one solution is $x=0$. For invertible matrices this is the only solution. For matrices not invertible there are nonzero solutions to $Ax=0$.

$A=\mathbb{R}^{m\times n}$

- HW 04:

prove $N(A)$ is a subspace of $\mathbb{R}^n$

→ $N(A)$ consists of all solutions $Ax=0$. Since A is an $m\times n$ matrix, to multiply A with x, x must be an $n\times 1$ vector. this means that x belongs to $\mathbb{R}^n$. Verify that $N(A)$ is a subspace: Suppose x and $y \in N(A)$. This means $Ax=0$ and $Ay=0$. Addition gives us $A(x+y)=0$. Scalar multiplication gives $A(cx)=c0$. $(x+y)$ and cx are $\in N(A)$. therefore $N(A)$ is a subspace.

* The column space consists of all linear combinations of the columns. the combinations are all possible vectors Ax. To get every possible b, we use every possible x. column space (is) is the vectors that can be written as A times some vector x.

inverse matrix $= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

determinant

- To solve $Ax=b$ is to express b as a combination of the columns.
- the system $Ax=b$ is solvable iff b is in the column space of A.

COLUMN SPACE OF A:

* ∘ column space of A contains all combinations of the columns A: a subspace of $\mathbb{R}^m$.

$A=\mathbb{R}^{m\times n}$ HW 04:

prove $C(A)$ is a subspace of $\mathbb{R}^m$

→ If A is an $m\times n$ matrix, the columns have m components, so the columns belong to $\mathbb{R}^m$. $C(A)$ is therefore a subspace of $\mathbb{R}^m$. Further show $C(A)$ is closed under addition and scalar multiplication. let $\vec{a}_1, \vec{a}_2, \dots \vec{a}_n$ be columns of A with b and $c \in col(A)$ so that $Ax=b$ and $Ay=c$. You have $A(x+y)=b+c$. Since b and c are linear combinations of the columns of A, then $b+c$ must still be a linear ~~function~~ combination. For scalar multiplication: $Ax=b$ where b is the col space of A. then $A(\alpha x) = \alpha x_1\vec{a}_1 + \dots + \alpha x_n \vec{a}_n$

$= \alpha(x_1\vec{a}_1 + \dots + x_n\vec{a}_n)$

$= \alpha b$ therefore closed under multiplication.

- If A has n columns then $\dim(col(A)) + \dim(Null(A)) = n$
  $rank(A) + \dim(Null(A))$
- If $col(A)$ and $col(A^T)$ are the same dimension then they span.
- eigenvalues of $AB \neq \lambda_A \lambda_B$
- eigenvalues of $AB = BA$

come back to later...

- col space of A: set of all x for which Ax is nonzero
- null space of A: set of all x for which $Ax=0$
- row space of A: set of all y for which $A^T y=0$
- left null space of A: set of all y for which $A^T y=0$

Def: a square matrix $A \in \mathbb{R}^{n\times n}$ is said to be symmetric if $A = A^T$

THEOREM: Real symmetric matrices have some formidable properties

① Any $A \in \mathbb{R}^{n\times n}$ with $A = A^T$ has ⓝ linearly independent eigenvectors
$\Rightarrow$ A is diagonizable

② the eigenvectors of A are mutually ORTHOGONAL (an orthogonal system of vectors)

③ All eigenvalues of A are REAL.

quadratic Functions on $\mathbb{R}^n$:

$q(\bar{x}) : \mathbb{R}^n \to \mathbb{R}$; $q(\bar{x}) = \frac{1}{2}\bar{x}^T A \bar{x} + \bar{b}^T x + c$ with $A \in \mathbb{R}^{n\times n}$; $A = A^T$
$\bar{b} \in \mathbb{R}^n$; $c \in \mathbb{R}$

THE GRAM-SCHMIDT ALGORITHM (ORTHOGONALIZATION)

- given a set of linearly independent vectors $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ compute an orthogonal system of vectors $\{\bar{e}_1, \bar{e}_2, \ldots \bar{e}_k\}$ such that

$\text{span}\{\bar{e}_1\} = \text{span}\{\bar{v}_1\}$
$\text{span}\{\bar{e}_1, \bar{e}_2\} = \text{span}\{\bar{v}_1, \bar{v}_2\}$
$\vdots$
$\text{span}\{\bar{e}_1, \bar{e}_2, \ldots \bar{e}_k\} = \text{span}\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$

$$\{\bar{v}_1, \bar{v}_2, \ldots \bar{v}_k\} \longrightarrow \boxed{\text{G.S.}} \longrightarrow \{\bar{e}_1, \bar{e}_2, \ldots \bar{e}_k\}$$

$\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$ is a basis for a given subspace
$\{\bar{e}_1, \bar{e}_2, \ldots \bar{e}_k\}$ is an orthogonal basis for the exact same subspace

STEP 1: $\vec{e}_1 \overset{\text{def}}{=} \vec{v}_1$ input data: ~~∅~~ the $\vec{v}$ vectors
output data: the $\vec{e}$ vectors

STEP 2: we take $\vec{e}_2$ to be of the form $\vec{e}_2 = \underbrace{\vec{v}_2}_{\text{input data}} + \underbrace{\alpha_{21}}_{\text{we need to compute}} \vec{e}_1 \to$ known from step 1

compute $\alpha_{21} \in \mathbb{R}$ from the fact that

$\bar{e}_2 \perp \bar{e}_1 : \langle \vec{e}_2, \vec{e}_1 \rangle = 0 \Leftrightarrow$

$\langle \vec{v}_2 + \alpha_{21}\vec{e}_1, \vec{e}_1 \rangle = 0 \Leftrightarrow \langle \vec{v}_2, \vec{e}_1 \rangle + \alpha_{21} \langle \vec{e}_1, \vec{e}_1 \rangle = 0 \Leftrightarrow \alpha_{21} = -\frac{\langle \vec{v}_2, \vec{e}_1 \rangle}{\langle \vec{e}_1, \vec{e}_1 \rangle}$

(note $\langle \vec{e}_1, \vec{e}_1 \rangle \neq 0$ since $\vec{e}_1 \neq \vec{0}$)

(check @ home that $\vec{e}_2 = \langle \vec{v}_2, \vec{e}_1 \rangle + \alpha_{21} \langle \vec{e}_1, \vec{e}_1 \rangle$ is orthogonal on $\vec{e}_1$) ?

STEP 3: compute $\vec{e}_3$ taken of the form $\vec{e}_3 = \underbrace{\vec{v}_3}_{\text{input data}} + \overbrace{\alpha_{23}}^{\text{to be computed}} \vec{e}_2 + \overbrace{\alpha_{13}}^{\text{to be computed}} \vec{e}_1$

$\vec{e}_2$, $\vec{e}_1$: known from steps 1 and 2.

uh... just check photos...

$\boxed{\text{check @ home that } \vec{e}_2 = \vec{v}_2 + \alpha_{21}\vec{e}_1 \text{) is orthogonal on } \vec{e}_1 \text{!}}$

Makes no sense

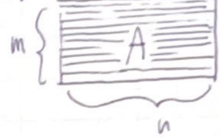
orthogonal projections

Def: A symmetric matrix $A \in \mathbb{R}^{m \times m}$ is said to be positive definite if $(\forall)\ \vec{x} \in \mathbb{R}^m$ it holds that $x^T A \vec{x} > 0$ and positive (semi)definite if $(\forall)\ \vec{x} \in \mathbb{R}^m$ it holds that $\vec{x}^T A \vec{x} \geq 0$.

- $A \in \mathbb{R}^{m \times m}$ symmetric has only real eigenvalues and m mutually orthogonal eigenvectors.

THEOREM: if you have $(\vec{x}^T A \vec{x})^T = \vec{x}^T A^T (\vec{x}^T)^T = \vec{x}^T A \vec{x}$ because $(ABC)^T = C^T B^T A^T$. inequality makes sence because A is symmetric and the L.H.S. turns into a scalar. $\Leftrightarrow$ A has strictly positive eigenvalues. $\Leftrightarrow$ A has non-strictly positive eigenvalues.

- Let $A \in \mathbb{R}^{m \times n}$ with $m < n$ having linearly independent rows. then (AA^T) is obviously symmetric and also invertible (AA^T is called the grammian of the rows of A)



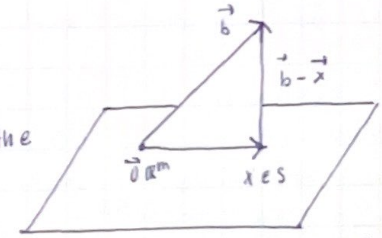
"Least Squares"

- let $A \in \mathbb{R}^{m \times n}$ with $m > n$ denote $S = C(A)$
  let $\vec{b} \in \mathbb{R}^m$ such that $\vec{b} \notin S$

- denote its columns $A = a_1, a_2, \dots a_n$ with the a's $\in \mathbb{R}^m$

Problem: Find the vector $\vec{x} \in S$ which best approximates $\vec{b} \notin S$

$\vec{r} \overset{def}{=} (\underbrace{\vec{b} - \vec{x}}_{\text{the residual}})$ is small $\Leftrightarrow$ length of $\vec{r} = \vec{b} - \vec{x}$ is small $\Leftrightarrow$ the square of the norm of $\vec{r} = \vec{b} - \vec{x}$ is small $\Leftrightarrow$ $\|\vec{b} - \vec{x}\|^2$ is small.



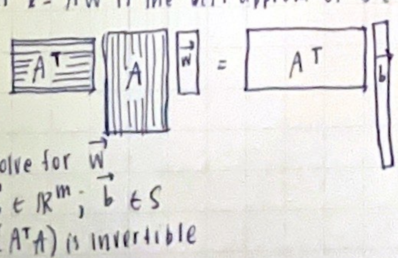
S is subspace of $\mathbb{R}^m$

THEOREM (ORTHOGONAL PROJECTION): $\vec{x} \in S$ is the best approximation of $\vec{b} \notin S$ if and only if the residual $(\vec{b} - \vec{x})$ is orthogonal on $S = C(A)$

DEFINITION: A vector $\vec{r}$ is said to be orthogonal on a hyperplane S iff $\vec{r}$ is orthogonal on any vector in S or equivalently iff $\vec{r}$ is orthogonal on each vector in some basis of S.

$$(\vec{b} - \vec{x}) \perp S \Leftrightarrow \begin{cases} \cdot\ \vec{b} - \vec{x} \perp \vec{a} \\ \cdot\ \vec{b} - \vec{x} \perp \vec{a}_2 \\ \vdots \\ \vec{b} - \vec{x} \perp \vec{a}_n \end{cases} \Leftrightarrow \begin{cases} (\vec{b} - \vec{x})^T \vec{a}_1 = 0 \\ (\vec{b} - \vec{x})^T \vec{a}_2 = 0 \\ \vdots \\ (\vec{b} - \vec{x})^T \vec{a}_n = 0 \end{cases} \Leftrightarrow (\vec{b} - \vec{x})^T A = [\underbrace{0\ 0 \dots 0}_{n \text{ times}}] \Leftrightarrow A^T(\vec{b} - \vec{x}) = \vec{0}\ \mathbb{R}^n$$

- $\vec{x}$ is the best approximation $\Leftrightarrow$ $A^T(\vec{b} - \vec{x}) = \vec{0}\ \mathbb{R}^n$. However, $\vec{x} \in S = C(A) \Leftrightarrow (\exists)\ \vec{w} \in \mathbb{R}^n$ such that $\boxed{\vec{x} = A\vec{w}}$.
  If $\vec{x} = A\vec{w}$ is the best approx. of $\vec{b} \in S$ then $A^T(\vec{b} - A\vec{w}) = \vec{0}\ \mathbb{R}^n \Leftrightarrow A^T\vec{b} - A^T A \vec{w} = \vec{0}\ \mathbb{R}^n \Leftrightarrow A^T A \vec{w} = A^T \vec{b}$



$\Leftrightarrow \boxed{\vec{w} = (A^T A)^{-1} A^T \vec{b}}$ $\quad \vec{x} = A\vec{w} = A(A^T A)^{-1} A^T \vec{b}$

unique solution.

- Solve for $\vec{w}$
- $\vec{b} \in \mathbb{R}^m$; $\vec{b} \in S$
- $(A^T A)$ is invertible

$\Longrightarrow$